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Magnetic deformation of ferrogel bodies: Procrustes effect

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Deformation of spheroidal ferrogel bodies caused by a uniform magnetic field is investigated theoretically. The deformation is induced by two competitive mechanisms—magnetostatic and magnetostrictive. The former is due to the demagnetizing field of the sample and hence depends on its shape, while the latter originates from the magnetoelasticity of ferrogel and is shape independent. Both mechanisms are dipolar in nature and contribute—for a body of commensurate dimensions—oppositely to the effect. For an isotropic ferrogel sphere, the magnetostatic contribution still prevails and the magnetic field elongates the body. The two opposing mechanisms balance each other out for a prolate spheroidal sample with the axes aspect ratio $a/b \approx 1.3$. It determines the so-called "Procrustes point" or "Procrustes size"—the magnetic field shrinks the body if $a > 1.3b$ and stretches it when $a < 1.3b$.

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I. INTRODUCTION

Ferrogels—also called magnetic or magnetorheological elastomers—are an important new class of materials consisting micro- or nanograins of a ferromagnet embedded firmly in a soft elastic polymeric matrix [1–6]. All existing and conceivable ferrogel applications (for example, as artificial muscles for robots) are based either on their large deformations achievable in magnetic fields even of a moderate intensity, or on the inverse effect: tensile or compressive strains sufficiently change the magnetic permeability of these materials, thereby enabling magnetic stress/strain sensors.

In view of the appearance of such smart materials, the old topic of magnetically induced mechanical stress [7–9] has once again attracted much attention [10–15]. Nonetheless, to date there is neither a theory capable of satisfactorily describing the magnetically induced strain, nor is there a convenient formula available for evaluating the effect.

To clarify mechanisms of magnetic deformation, consider an ellipsoid of revolution whose axis of symmetry (axis x) is aligned with a uniform applied field $\boldsymbol{\mathcal{H}}$. There are two causes giving rise to deformation of the body by the field; both of them enhance the body magnetization $\mathbf{M}$ diminishing the magnetic part of its energy [7],

$$U_{\mathrm{mag}} = -\frac{V}{2}(\mathbf{M}\cdot\boldsymbol{\mathcal{H}}) = -\frac{V\mathcal{H}^2}{8\pi}\frac{\mu^{(x)}-1}{1+n(\mu^{(x)}-1)}. \tag{1}$$

Here n and $\mu^{(x)}$ are the demagnetizing factor and the magnetic permeability of the body in the field direction, and V is the body volume. One of the causes of body deformation is due to *magnetostatics*: the sample tends to elongate along the field to decrease n in the denominator of Eq. (1) in order to minimize the demagnetizing field opposing the applied one. So, this effect (providing by Maxwell's stress) depends strongly on the body shape.

Another reason for the body strain is *magnetostriction*: the sample changes its length in the direction of the field to enhance the magnetic permeability $\mu^{(x)}$. In fact, polarizing embedded magnetic grains, an imposed field switches on dipolar (i.e., anisotropic) interaction between them. The latter manifests itself as an internal stress. In the case of random distribution of spherical magnetic grains in an initially isotropic elastic matrix this stress compresses the body along the field and stretches it out in the transverse direction. It results in the replacement of the scalar magnetic permeability μ_0 by the tensor [7],

$$\mu_{ik} = \mu_0\delta_{ik} + a_1u_{ik} + a_2u_{ll}\delta_{ik}, \tag{2}$$

where a_1 and a_2 are material parameters, u_{ik} is the strain tensor, and $u_{ll} = \mathrm{div}\,\mathbf{u}$ is the divergence of the displacement vector. The strain is assumed to be small and then only first-order terms in u_{ik} are retained in Eq. (2). This expression describes also an *inverse effect*: an applied mechanical stress changes the magnetic permeability in accordance with the direction of the force.

II. SPHERE: ENERGY APPROACH

Half a century ago [7], Landau and Lifshitz (L&L) determined the change in shape of a dielectric sphere ($n=1/3$) under uniform electric field. Their result is given below in terms of the magnetic (isomorphic) problem.

According to [7], the sphere of the radius R is changed into spheroid of the same volume V (i.e., $u_{ll}=0$) with semiaxes $a \neq b = c$. Moreover, L&L assumed the strain to be a uniform pure shear in the volume of the body: $\varepsilon \equiv u_{xx} = (a/R) - 1$. Substituting in Eq. (1),

$$n = \tfrac{1}{3} - \tfrac{2}{5}\varepsilon, \quad \mu^{(x)} = \mu_0 + a_1\varepsilon,$$

and minimizing the sum of the magnetic [Eq. (1)] and elastic $U_{\mathrm{el}} = (3/2)GV\varepsilon^2$ energies (G stands for the shear modulus), L&L have found the body elongation,

$$\varepsilon = \frac{3\mathcal{H}^2}{8\pi G}\left(\frac{\mu_0-1}{\mu_0+2}\right)^2\left[\frac{2}{5} + \frac{a_1}{(\mu_0-1)^2}\right]. \tag{3}$$

It is convenient to rewrite this expression introducing the magnetization M_0 which would belong to the undeformable sphere with magnetic permeability μ_0,

$$\varepsilon = \frac{2\pi M_0^2}{3G}\left[\frac{2}{5} + \frac{a_1}{(\mu_0-1)^2}\right], \quad M_0 = \frac{3\mathcal{H}}{4\pi}\frac{\mu_0-1}{\mu_0+2}. \tag{4}$$

The terms in square brackets correspond to the shape and magnetostrictive contributions, respectively. As shown below, both of them are equally important. The shape effect still can be minimized by using elongated samples [14], but the magnetostriction cannot be avoided [16].

Calculation of magnetostriction parameters exceeds the limits of the phenomenological description given in Ref. [7]. Coefficients a_1 and a_2 can be expressed in terms of the "unperturbed" magnetic permeability μ_0 using microscopic considerations. For a simple-cubic lattice of spherical magnetically polarized inclusions in an isotropic nonmagnetic matrix, the problem was solved in Refs. [9,10]. For the case of random distribution of such grains, Shkel and Klingenberg have obtained [10]

$$a_1 = -\tfrac{2}{5}(\mu_0-1)^2,$$

$$a_2 = -\tfrac{1}{3}(\mu_0-1)(\mu_0+2) + \tfrac{2}{15}(\mu_0-1)^2. \tag{5}$$

Substituting the value a_1 in Eq. (4) gives $\varepsilon=0$: magnetostatic tension and magnetostrictive compression [17] equilibrate each other, thus the magnetoelastic sphere remains *undeformed* under the field.

The situation changes if the original body's shape even weakly deviates from spherical one. Then $n=(1/3)+\Delta n$, and the field-induced deformation reduces to

$$\varepsilon = \frac{8\pi M_0^2}{5G}\frac{\mu_0-1}{\mu_0+2}\Delta n. \tag{6}$$

As seen, the applied field "fits" the sample to a sphere: it shrinks the prolate spheroid, $\Delta n<0$, and stretches out the oblate one, $\Delta n>0$. Thus the competition between magnetostatics and magnetostriction results in the *Procrustes effect*.

III. SPHERE: EXACT SOLUTION

Thus, seemingly just the sphere is a candidate for the "Procrustes bed." However, one should remember that the L&L relationship [Eq. (3)] gives only a rather rough estimate of the effect. The foregoing result was found in the frame of the energy approach under the strong and far from obvious assumption about uniformity of the field-induced strain. However, there is no necessity to predetermine the character of the strain. Fortunately, the equations of elasticity and corresponding boundary conditions at the sphere surface allow an exact solution to the problem in the full-scale formulation.

The stress tensor inside the sphere consists of magnetic [7] and elastic [18] parts, $\sigma_{ik}^{\text{in}}=\sigma_{ik}^{\text{mag}}+\sigma_{ik}^{\text{el}}$, where

$$\sigma_{ik}^{\text{mag}} = \frac{2\mu_0-a_1}{8\pi}H_iH_k - \frac{\mu_0+a_2}{8\pi}H^2\delta_{ik}, \tag{7}$$

$$\sigma_{ik}^{\text{el}} = 2G\left(u_{ik} + \frac{\nu}{1-2\nu}u_{ll}\delta_{ik}\right). \tag{8}$$

We introduce here the Poisson ratio ν, and the internal field $\mathbf{H}$ linked with the magnetization by the relation $\mathbf{M}_0=(\mu_0-1)\mathbf{H}/4\pi$—cf. Eq. (4). Stresses outside the sphere are described by usual Maxwell's stress tensor for vacuum. At the sphere surface, $r=R$, the boundary conditions of continuity of the normal and tangential stresses should be satisfied. With allowance for continuity of the normal component of the induction $\mu_0 H_r$ and the tangential component of the field H_θ, one finds

$$\sigma_{rr}^{\text{el}}|_R = 2\pi M_0^2[A_2 + (1+A_1)\cos^2\theta], \tag{9}$$

$$\sigma_{r\theta}^{\text{el}}|_R = -2\pi M_0^2 A_1 \sin\theta\cos\theta, \tag{10}$$

where θ is the polar angle, A_1 and A_2 stand for the rescaled magnetostriction parameters [Eq. (5)],

$$A_1 = \frac{a_1}{(\mu_0-1)^2}, \quad A_2 = \frac{1}{\mu_0-1} + \frac{a_2}{(\mu_0-1)^2}. \tag{11}$$

The equation describing an equilibrium of a ferrogel sphere in a uniform magnetic field reads [18]

$$(1-2\nu)\Delta\mathbf{u} + \nabla\text{div}\,\mathbf{u} = 0. \tag{12}$$

The uniform magnetic field enters into the displacement vector $\mathbf{u}$ only through the boundary conditions [Eqs. (9) and (10)] on the unperturbed surface of the sphere. Satisfying these conditions and using Eq. (8), we find both components of the vector $\mathbf{u}$, (its azimuthal component is absent because of the axial symmetry of the problem),

$$u_r = u_{\text{cc}} + \frac{2\pi M_0^2}{3G}\left[\frac{7+2\nu}{7+5\nu} + A_1 - \frac{6\nu}{7+5\nu}\left(\frac{r}{R}\right)^2\right] rP_2(\cos\theta), \tag{13}$$

$$\mathbf{u}_{\text{cc}} = \frac{\pi M_0^2}{3G}\frac{1-2\nu}{1+\nu}(1+A_1+3A_2)\mathbf{r}, \tag{14}$$

$$u_\theta = \frac{\pi M_0^2}{3G}\left[\frac{7+2\nu}{7+5\nu} + A_1 - \frac{7-4\nu}{7+5\nu}\left(\frac{r}{R}\right)^2\right] rP_2'(\cos\theta). \tag{15}$$

Here the spherically symmetric part of the solution, $\mathbf{u}_{\text{cc}}$, describes the *comprehensive compression/expansion* of the body; P_2 is the Legendre polynomial, and prime in Eq. (15) denotes the derivative over the polar angle.

The relative change in the volume of the sphere exposed to uniform magnetic field is yielded by Eq. (14),

$$\frac{\Delta V}{V} = \frac{\pi M_0^2}{G}\frac{1-2\nu}{1+\nu}(1+A_1+3A_2). \tag{16}$$

Substituting here A_1 and A_2 from Eq. (11) gives $\Delta V=0$, i.e., leads us to a surprising conclusion: an isotropic magnetoelastic sphere conserves its volume in a uniform magnetic field. Notably, this conclusion does not depend on the value of Poisson ratio ν.

Thus, the change in shape of the sphere is provided by pure—but nonuniform—shear strains. For this case it is natural to determine the elongation parameter ε as a ratio of the radial displacement of the sphere's pole to the initial radius of the sphere,

$$\varepsilon = \frac{u_r(r=R,\theta=0)}{R}. \tag{17}$$

Substituting here Eqs. (11) and (13) results in

$$\varepsilon = \frac{2\pi M_0^2}{5G}\frac{7-10\nu}{7+5\nu} \tag{18}$$

instead of $\varepsilon=0$ found earlier in the energy approximation. In the incompressible limit, $\nu=0.5$, Eq. (18) gives

$$\varepsilon = (8\pi/95)M_0^2/G, \tag{19}$$

which is only 24% of the elongation that would be attained in the absence of magnetostriction [15]. Nonetheless, ε still remains positive, so the shape effect predominates over the magnetostriction for the spherical sample.

IV. SPHEROID: PROCRUSTES POINT

Replace the sphere with a prolate spheroid of aspect ratio $L=a/b>1$; its long axis is aligned with the magnetic field. An increase in L leads to some decrease in the demagnetizing factor n in the field direction, and for $L\gg 1$ the shape effect becomes negligible. Then it remains only magnetostrictive compression along the field. Taking into account both of the cases—stretching of a sphere and compressing of a long cylinderlike body—one should expect *changing the sign* of elongation ε at a certain aspect ratio L_P. We call it the *Procrustes point* or *Procrustes size* in accordance with mentioned above argumentation. The shape effect prevails at $L<L_P$ whereas magnetostriction dominates at $L>L_P$. When $L=L_P$, both counteracting tendencies get balanced.

To investigate magnetodeformation of a spheroid, we apply Eqs. (8) and (12), and boundary conditions for stresses on its surface. Introducing the surface normal (**n**) and tangential (τ) vectors and the unit vector **h** along the external field, we get

$$\sigma_{nn}^{\rm el} = \tfrac{2}{3}\pi M_0^2[1+A_1+3A_2+2(1+A_1)P_2(\mathbf{nh})], \tag{20}$$

$$\sigma_{n\tau}^{\rm el} = \tfrac{2}{3}\pi M_0^2 A_1 P_2'(\mathbf{nh}). \tag{21}$$

These boundary conditions generalize the relations (9) and (10) given above for spherical bodies. As seen from Eq. (20), the spherically symmetric part of the normal stress is proportional to the value $(1+A_1+3A_2)$ which is equal to zero [Eq. (16)]. This however does not signify that the *volume* of an ellipsoidal body remains unchanged. The obtaining radial displacement u_r turns out to be proportional to $P_2(\mathbf{nh})$—cf. Eq. (13). Subsequent integrating u_r over the surface of the body results in $\Delta V=0$ only for a sphere where the angle between vectors **n** and **h** coincides with the polar angle θ. But in the case of a spheroid $(L\neq 1)$ the integral differs from zero.

The solution to the problem we have found in the form of a very awkward series over the Legendre polynomials $P_{2k}(\mathbf{nh})$. A simple approximate solution is obtained if only the lowest-order term with $k=1$, is retained in the series. In this way, for the elongation parameter of an incompressible body, we obtain the interpolation formula,

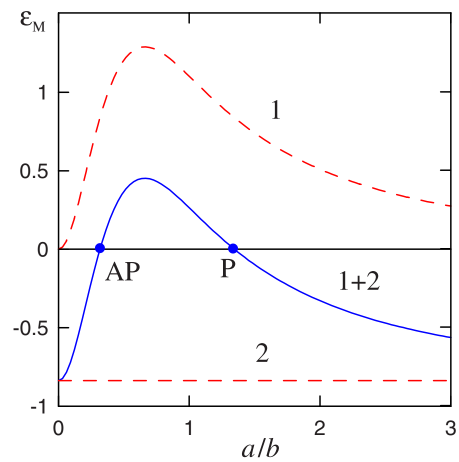

FIG. 1. (Color online) Contributions of the demagnetizing field (curve 1) and the magnetostriction (line 2) to the resulting elongation (curve 1+2) of a spheroidal body vs its aspect ratio. The reduced elongation, $\varepsilon_M=(G/M_0^2)\varepsilon$, is calculated for an incompressible body (Poisson's ratio $\nu=0.5$).

$$\varepsilon = \frac{4\pi M_0^2}{15G}\left[\frac{175L^2(7+10L^2+8L^4)}{(3+24L^2+8L^4)(23+24L^2+48L^4)}-1\right], \tag{22}$$

where the first (positive) term describes the demagnetizing field effect and the second (negative) term corresponds to the magnetostriction. Both these contributions and their sum are depicted in Fig. 1. Note that the approximate dependence [Eq. (22)] reproduces *exact results* for a sphere, $L=1$, a long cylinder, $L\to\infty$, and a thin disk, $L\to 0$. Indeed, for $L=1$ this relation gives the result [Eq. (19)], while for two pointed above limiting cases, Eq. (22) yields the same shrink,

$$\varepsilon = -(4\pi/15)M_0^2/G. \tag{23}$$

Two nodes of the function $\varepsilon(L)$ correspond to the Procrustes', $L_{\rm P}=1.345$, and anti-Procrustes', $L_{\rm AP}=0.317$, points. The latter expresses the fact that magnetostatics is ineffective in stretching of *flat* incompressible samples—see the upper curve in Fig. 1. Therefore magnetostriction predominates over magnetostatics not only for $L>L_{\rm P}$ but when $L<L_{\rm AP}$, too. The situation changes dramatically for compressible objects. As seen in Fig. 2, the magnitude of $L_{\rm AP}$ decreases rapidly with a decrease in Poisson's ratio ν and disappears at $\nu=0.25$. Meanwhile $L_{\rm P}$ value changes insignificantly: from 1.345 to 1.519.

V. STRAIN'S ESTIMATE

The reduced elongation $\varepsilon_M(L)$ shown in Figs. 1 and 2 is related to the real one $\varepsilon(L)$ by the ratio $\varepsilon=(M_0^2/G)\varepsilon_M$. However, the body magnetization M_0 itself is a function of L. Then the external field $\mathcal{H}$ but not M_0 is the true control parameter. Corresponding dependence denoted as $\varepsilon_H(L)$ is shown in Fig. 3. For a soft incompressible magnetic elastomer of $G=3$ kPa (Young's modulus=9 kPa) and $\mu_0=3$, in the field $\mathcal{H}=800$ Oe one obtains $\varepsilon=+5.1\%$ for a spherical body and $\varepsilon=-10.3\%$ for a prolate spheroid with $L=2$.

RAPID COMMUNICATIONS

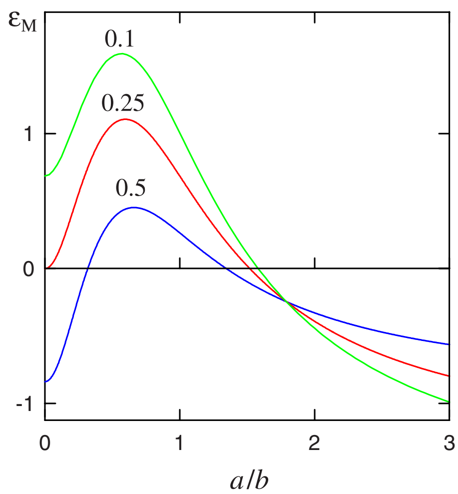

FIG. 2. (Color online) The reduced resulting elongation as defined in Fig. 1 versus the aspect ratio $L=a/b$ for three magnitudes of the Poisson's coefficient: $\nu=0.50,0.25,0.10$.

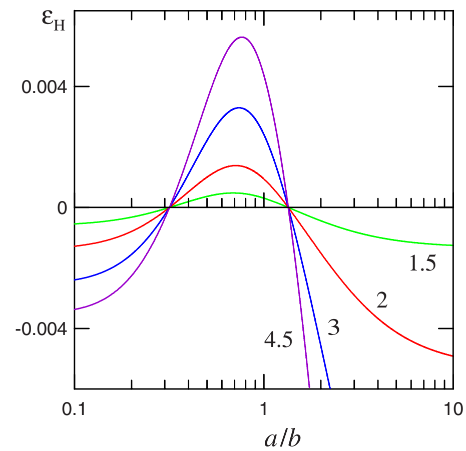

FIG. 3. (Color online) Reduced elongation, $\varepsilon_H=(G/\mathcal{H}^2)\varepsilon$, of an incompressible spheroidal sample versus its aspect ratio $L=a/b$ for some values of its magnetic permeability μ_0.

VI. CONCLUSIONS

An external uniform magnetic field deforms a ferrogel sample. The field tends to adjust an ellipsoidal body to the certain shape. Prolate spheroid with the aspect ratio $L=a/b$ compresses in the field direction if L exceeds some characteristic—"Procrustes"—value $L_P \simeq 1.35$. Contrarily, the body elongates when L is less than L_P. At the Procrustes point, $L=L_P$, the competing effects of the demagnetizing field and magnetostriction just balance to produce no shape change.

ACKNOWLEDGMENTS

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[16] The authors of Ref. [15] have overlooked the magnetostriction. They ignored it by setting a_1 and a_2 to zero and considered only the demagnetizing field effect.
[17] As seen from Eq. (2), if $a_1<0$ just the compression (i.e., the negative strain) leads to an increase in μ and hence to some decrease in the magnetic free energy [Eq. (1)].
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